

SYMBOL LENGTH IN THE BRAUER GROUP OF A FIELD

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ABSTRACT. We bound the symbol length of elements in the Brauer group of a field K containing a C_m field (for example any field containing an algebraically closed field or a finite field), and solve the local exponent-index problem for a C_m field F . In particular, for a C_m field F , we show that every F central simple algebra of exponent p^t is similar to the tensor product of at most $\text{len}(p^t, F) \leq t(p^{m-1} - 1)$ symbol algebras of degree p^t . We then use this bound on the symbol length to show that the index of such algebras is bounded by $(p^t)^{(p^{m-1}-1)}$, which in turn gives a bound for any algebra of exponent n via the primary decomposition. Finally for a field K containing a C_m field F , we show that every F central simple algebra of exponent p^t and degree p^s is similar to the tensor product of at most $\text{len}(p^t, p^s, K) \leq \text{len}(p^t, L)$ symbol algebras of degree p^t , where L is a $C_{m+\text{ed}_L(A)+p^s-t-1}$ field.

1. INTRODUCTION

We are interested in the following two problems:

The symbol length problem: Let F be a field and A a F central simple algebra of exponent n . Assuming F contains a primitive n -th root of one ρ_n , the Merkurjev-Suslin theorem tells us that any such A is brauer equivalent to the tensor product of symbol algebras of degree n . The minimal number of symbol algebras needed is called the symbol length of A denoted $\text{len}(n, A)$. The symbol length problem asks if there is a finite bound $\text{len}(n, F)$, such that for any $A \in \text{Br}(F)$ of exponent n one has $\text{len}(n, A) \leq \text{len}(n, F)$. One can filter the n -th torsion of the brauer group by degree and define $\text{len}(n, m, F)$ or $\text{len}(n, m)$ when it is independent of the field F , as the minimal number of symbols needed to express every $A \in \text{Br}(F)$ of exponent n and degree m . Notice that

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the existence of a generic division algebra of exponent n and degree m implies $\text{len}(n, m)$ is always finite. Finding an explicit bound for $\text{len}(n, m)$ is also referred to as the symbol length problem.

The exponent-index problem: It is well known that for any $A \in \text{Br}(F)$ the exponent of A divides the index of A and the two numbers have the same prime factors; in particular the exponent is bounded by the index. The exponent-index problem asks if one can bound the index in terms of the exponent. To be more precise, for a prime p define the Brauer dimension at p , denoted $\text{Br. dim}_p(F)$, to be the smallest integer d such that for all $n \in \mathbb{N}$ and $A \in \text{Br}_{p^n}(F)$, $\text{ind}(A)$ divides $\exp(A)^d$, and ∞ if no such number exists. Then define the Brauer dimension of F to be $\text{Br. dim}(F) = \sup\{\text{Br. dim}_p(F)\}$. The global exponent-index problem asks if $\text{Br. dim}(F)$ is finite and the local exponent-index problem asks if $\text{Br. dim}_p(F)$ is finite.

The answer to these problems is negative for arbitrary fields. To see this consider the field $F = \mathbb{Q}[\rho_p](x_1, y_1, \dots, x_i, y_i, \dots)$ and define $A_n = \otimes_{i=1}^n (x_i, y_i)_{F, p}$. Then it is known that A_n is a division algebra (see for example [12, Corollary 1.2]), that is $\text{ind}(A_n) = p^n$, and $\exp(A_n) = p$. In particular $\text{len}(p, A_n) = n$ and $\text{Br. dim}_p(F) \geq n$ for all $n \in \mathbb{N}$, implying $\text{len}(p, F) = \infty$ and $\text{Br. dim}(F) = \text{Br. dim}_p(F) = \infty$.

It seems that a positive answer to these problems is strongly related to the arithmetic of the base field F . This is supported by the following results:

- (1) For F a local or global field, $\text{Br. dim}(F) = 1$ by the Albert-Brauer-Hasse-Noether theorem [2] and [8].
- (2) For F a C_2 field, M. Artin conjectured [3] that $\text{Br. dim}(F) = 1$. He proved that $\text{Br. dim}_2(F) = \text{Br. dim}_3(F) = 1$ for such fields.
- (3) For F a finitely generated field of transcendence degree 2 over an algebraically closed field, $\text{Br. dim}(F) = 1$ by [12] and [14, Theorem 4.2.2.3].
- (4) For F finitely generated and of transcendence degree 1 over an ℓ -adic field, $\text{Br. dim}_p(F) = 2$ for every prime $p \neq \ell$ by [23].

Motivated by M. Artin's results ([3]) we focus our attention on a class of fields called C_m fields. A field F is called C_m if it has the property that every homogeneous equation $f(x_1, \dots, x_n) = 0$, of degree d has a non-trivial solution when $n > d^m$.

We solve both the symbol length and the local exponent-index problems for these fields by giving explicit bounds on $\text{len}(n, F)$ and $\text{Br. dim}_p(F)$. In particular we prove the following theorems,

Theorem 4.4 Let F be a C_m field containing all n -th roots of unity and $A \in \text{Br}_n(F)$ be of exponent $n = p^t$. Then: $A \sim \otimes_{i=1}^t C_i$ where $C_i = \otimes_{j=1}^{p^{m-1}-1} (\alpha_{i,j}, \beta_{i,j})_{p^i}$. In particular $\text{len}(p^t, F) \leq t(p^{m-1} - 1)$.

Theorem 5.3 Let F be a field containing a C_m field L and all p^t -th roots of unity, and A be a F -csa of exponent p^t and degree p^s . Then the symbol length of A is bounded by $\text{len}(p^t, K)$ where K is a $C_{m+\text{ed}_L(A)+p^{s-t}-1}$ field.

Theorem 6.3 If F is a C_m field, then $\text{Br. dim}_p(F) \leq p^{m-1} - 1$.

Theorem 8.2 Let F be a C_m field and let $\alpha \in K_2^M(F)/nK_2^M(F)$ where $n = p^t$, then α can be written as the sum of at most $t(p^{m-1} - 1)$ symbols.

The approach we take is to first bound the symbol length and then use this bound to get a bound for $\text{Br. dim}_p(F)$. To bound the symbol length we start with $A \in \text{Br}_{p^n}(F)$, use the Merkurjev-Suslin theorem to assume it is a product of symbol algebras, and then show how to shorten the number of symbol algebras down to a fixed number. The key idea is to consider $A_k = \otimes_{i=1}^k (\alpha_i, \beta_i)$ for $k \in \mathbb{N}$ and produce “large” vector spaces $V_k \leq A_k$ called n -Kummer spaces with the property that for every $v \in V_k$ one has $v^n \in F$. These spaces have “low” degree norm forms, $N_i : V_i \rightarrow F$ defined by $N_i(v) = v^n$. Thus when k is big enough the C_m property ensures the existence of a non trivial solution for $N_k(v) = 0$ from which we deduce how to shorten the number of symbols.

The paper is organized as follows: We start with a background section where we give the main definitions needed for this work and some known results in the subject. In section 3 we use n -Kummer spaces and their norm forms to get bounds on the symbol length for arbitrary exponent n . In section 4 we use known results about primary decomposition in the Brauer group and a divisibility property of symbol algebras to improve the bounds obtained in section 3. In section 5 we generalize the discussion to fields containing a C_m for some m . Section 6 is devoted to the exponent-index problem, where we use the results in section 4 to solve the local exponent-index problem. Section 7 is devoted to the characteristic $p > 0$ case. Finally in the section 8 we

show that our results can be formulated in the context of the second Milnor K -group where the presence of roots of unity is not required.

2. BACKGROUND

2.1. The Brauer group. Let F be a field. A F -central simple algebra, denoted F -csa, is an F -algebra, simple as a ring with center F . The Brauer group of F is defined as

$$\{\text{isomorphism classes of finite dimensional } F\text{-csa}\} / \sim$$

where for two F -csa A and B ,

$$A \sim B \Leftrightarrow \exists n, m \in \mathbb{N} : M_n(A) \cong M_m(B)$$

It is well-known that $\text{Br}(F)$ is a torsion group. We write $\exp(A)$ for the order of A in $\text{Br}(F)$ and $\text{Br}_n(F)$ for the n -torsion subgroup of $\text{Br}(F)$. By the Wedderburn-Artin theorem every F -csa A is isomorphic to $M_n(D)$ for unique $n \in \mathbb{N}$ and F -central division algebra D called the underlying division algebra of A . One defines the degree and index of A as $\deg(A) = \sqrt{\dim_F(A)}$ and $\text{ind}(A) = \deg(D)$ respectively. For a more detailed study of the Brauer group we refer the reader to [21], [11] or [7].

An important example of F -csa are symbol algebras which we now define. Let F be a field containing a primitive n -th root of 1 denoted ρ_n and $a, b \in F^\times$. Define the symbol algebra

$$(a, b)_{n,F} = F[x, y | x^n = a, y^n = b, yx = \rho_n xy].$$

Then $(a, b)_{n,F}$ is a F -csa of degree n and exponent dividing n .

A standard pair of generators for $(a, b)_{n,F}$ is a pair $u, v \in (a, b)_{n,F}$ satisfying $u^n \in F^\times$, $v^n \in f^\times$ and $uv = \rho_n vu$, and for any such pair one has $(a, b)_{n,F} \cong (v^n, u^n)_{n,F}$.

The most famous example is the well known quaternion algebra which has the presentation $(-1, -1)_{2, \mathbb{R}}$. The presentation of a F -csa as a symbol algebra (or the tensor product of several symbol algebras) is not unique, and starting from a given presentation one can produce many others. The following proposition tells us that we may assume one of the slots represents a field, that is,

Proposition 2.1. *Assume $A = (a, b)_n$ does not split. Then we can modify the presentation of A such that $F[x]$ is a field where $x^n = a$.*

Proof. It is enough to consider the case where $n = p^m$. Now $K = F[x]$ is a field if and only if $x^n - a$ is an irreducible polynomial if and only if $a \notin F^{\times p}$. Let $s = \max\{s | a \in F^{\times p^s}\}$, which is finite because $a \notin F^{\times p^m}$, as A does not split. Write $a = c^{p^s}$. Then by the definition of s we know $c \notin F^{\times p}$. Now write $A = (a, b)_n = (c^{p^s}, b)_n = (c, b^{p^s})_{n, F}$ and now $K = F[x | x^n = c]$ is a field. \square

We give some well known relations which will be used to prove the main theorems of this work.

Proposition 2.2. *Let $A_i = (a_i, b_i)_n$ with standard generators x_i, y_i , $i = 1, 2$, and let $N_{F[x_i]/F}$ denote the regular field norm.*

- (1) *For every $k_1 \in F[x_1]^\times$, $A_1 \cong (a_1, N_{F[x_1]/F}(k_1)b_1)$.*
- (2) *If $a_1 + b_1 \neq 0$ then, $A_1 \cong (a_1 + b_1, -a_1^{-1}b_1)_n$.*
- (3) *$A_1 \otimes A_2 \cong (a_1, b_1b_2^{-1})_n \otimes (a_1a_2, b_2)_n$.*
- (4) *For $k_2 \in F[x_2]^\times$, if $t = a_1a_2 + N_{F[x_2]/F}(k_2)b_2 \neq 0$ then $A_1 \otimes A_2 \cong (a_1, *)_n \otimes (t, *)_n$.*

Proof.

(1)+(2) are standard relations which can be found in [21], [11] or [7].

(3) Consider the commuting pairs

$$u_1 = x_1, v_1 = y_2^{-1}y_1 \text{ and } u_2 = x_1x_2, v_2 = y_2,$$

noting that $u_1^n = a_1$, $v_1^n = b_2^{-1}b_1$, $u_2^n = a_1a_2$, $v_2^n = b_2$.

(4) Combining (1), (2) and (3) we have,

$$\begin{aligned} A_1 \otimes A_2 &= (a_1, b_1)_n \otimes (a_2, b_2)_n \\ &\stackrel{(1)}{\cong} (a_1, b_1)_n \otimes (a_2, N_{F[x_2]/F}(k_2)b_2)_n \\ &\stackrel{(3)}{\cong} (a_1, b_1(N_{F[x_2]/F}(k_2)b_2)^{-1})_n \otimes (a_1a_2, N_{F[x_2]/F}(k_2)b_2)_n \\ &\stackrel{(2)}{\cong} (a_1, b_1(N_{F[x_2]/F}(k_2)b_2)^{-1})_n \otimes (a_1a_2 + N_{F[x_2]/F}(k_2)b_2, -(a_1a_2)^{-1}N_{F[x_2]/F}(k_2)b_2)_n \\ &= (a_1, *)_n \otimes (t, *)_n \end{aligned}$$

as claimed. \square

2.2. Severi-Brauer Varieties. Let A be a F -csa of degree n . The Severi-Brauer variety associated to A denoted $SB(A)$, is the variety of all minimal left ideals of A . The dimension of $SB(A)$ is $n - 1$.

This variety contains the splitting information for A as seen from the following theorem:

Theorem 2.3. ([24, Theorem 13.7]) *Let A be as above. The following are equivalent:*

- (1) $A \cong M_n(F)$, i.e. A is split.
- (2) $SB(A) \cong \mathbb{P}^{n-1}(F)$.
- (3) $SB(A)$ has a rational point.

One more important property of this variety is that its function field serves as a generic splitting field for A in the following sense:

Theorem 2.4. ([24, 13]) *The following are equivalent:*

- (1) K is a splitting field for A .
- (2) There is a place $\nu : F(SB(A)) \rightarrow K$.

For more on this important variety we refer the reader to [24] or [7].

2.3. C_m fields. Even though the definition of a C_m field seems quite restrictive there are many interesting fields which are C_m . Here are some known examples:

- (1) Every algebraically closed field is C_0 .
- (2) Every finite field is C_1 .
- (3) If F is C_m and $F \subset K$ is of transcendence degree n over F , then K is C_{m+n} , by [13] completed by [19].
- (4) The above implies that if V is a variety of dimension n over an algebraically closed field F , then the function field, $F(V)$, is C_n .

2.4. Known results. Results on symbol length:

- (1) Every algebra of degree 2 is isomorphic to a quaternion algebra. That is, $\text{len}(2, 2) = 1$.
- (2) Every algebra of degree 3 is cyclic and thus, when $\rho_3 \in F$ it is isomorphic to a symbol algebra. That is, $\text{len}(3, 3) = 1$ (Wedderburn [28]).
- (3) Every algebra of degree 4 of exponent 2 over a field of characteristic different from 2 is isomorphic to a product of two quaternion algebras. That is, $\text{len}(4, 2) = 2$ (Albert [1]).
- (4) Every algebra of degree 8 and exponent 2 is similar to the product of four quaternion algebras. That is, $\text{len}(8, 2) = 4$ (Tignol [27]).

- (5) Every algebra of degree 9 and exponent 9 over a field of characteristic different than 3 containing ρ_9 is similar to the product of 35840 symbol algebras of degree 9 and if it is of exponent 3 it is similar to the product of 277760 symbol algebras of degree 3. That is, $\text{len}(9, 9) \leq 35840$ and $\text{len}(9, 3) \leq 277760$ (Matzri [16]).
- (6) Every algebra of prime degree p over a field of characteristic different from p containing ρ_p is similar to the tensor product of $\frac{(p-1)!}{2}$ symbol algebras. That is, $\text{len}(p, p) \leq \frac{(p-1)!}{2}$ (Rosset-Tate [20], [7, 7.4.11] and Rowen-Saltman [22]).
- (7) Every p -algebra of index p^n and exponent p^m is similar to the product of $p^n - 1$ cyclic algebras of degree p^m . That is, $\text{len}(p^m, p^n) = p^n - 1$ (Florence [9]).
- (8) If F is the function field of an l -adic curve containing a primitive p -th root of one and p is a prime different than l , then every degree p algebra is cyclic. That is, $\text{len}(p, p, F) = 1$ (Saltman [23]).
- (9) If F is a local or global field containing a primitive n -th root of 1, every algebra of exponent n is a symbol. That is $\text{len}(n, F) = 1$ (Albert-Brauer-Hasse-Noether [2] and [8]).
- (10) If F is a C_2 field containing a necessary primitive root of 1 then, $\text{len}(2, F) = \text{len}(3, F) = 1$ (Artin [3]).
- (11) If F is the function field of an l -adic curve and $(n, l) = 1$, then every algebra of exponent n is the product of two cyclic algebras. Thus if F contains a primitive n -th root of 1, $\text{len}(n, F) = 2$ (Brussel, McKinnie and Tengan [5]).

Results on the exponent-index problem:

- (1) For F a local or global field, $\text{Br. dim}(F) = 1$ (Albert-Brauer-Hasse-Noether [2] and [8]).
- (2) For F a finitely generated of transcendence degree 2 over an algebraically closed field, $\text{Br. dim}(F) = 1$ (de-Jong [12] and Lieblich [14, Theorem 4.2.2.3]).
- (3) M. Artin conjectured in [3] that $\text{Br. dim}(F) = 1$ for every C_2 field F . He proved that $\text{Br. dim}_2(F) = \text{Br. dim}_3(F) = 1$ for such fields.

- (4) For F finitely generated and of transcendence degree 1 over an ℓ -adic field, $\text{Br. dim}_p(F) = 2$ for every prime $p \neq \ell$ (Saltman [23]).
- (5) If F is a complete discretely valued field with residue field k such that $\text{Br. dim}_p(k) \leq d$ for all primes $p \neq \text{char}(k)$, then $\text{Br. dim}_p(F) \leq d + 1$ for all $p \neq \text{char}(k)$ (Harbater, Hartmann and Krashen [10, Theorem 5.5]).
- (6) If F has characteristic p and is finitely generated of transcendence degree r over a perfect field k , then $\text{Br. dim}_p(F) \leq r$, by methods of Albert ([4, page 7]).

3. USING n -KUMMER SPACES TO BOUND THE SYMBOL LENGTH

As above, for $a, b \in F^\times$, we denote by $(a, b)_n$ the symbol algebra $F[x, y \mid x^n = a, y^n = b, yx = \rho xy]$.

We say that an F -subspace V of a central simple algebra A is n -Kummer if $v^n \in F$ for every $0 \neq v \in V$. An n -Kummer space V is endowed with the exponentiation map

$$N_V : V \rightarrow F, \text{ defined by } N_V(v) = v^n$$

which is a homogeneous form of degree n .

Remark 3.1.

- (1) If $\deg(A) = nm$, then $N_V(v)^m = \text{Nrd}_A(v)$.
- (2) An element $d \in A$ of degree n has characteristic polynomial $\lambda^n - \alpha$ (thus satisfies $d^n \in F$) if and only if $\text{tr}(d^m) = 0$ for all $1 \leq m \leq n-1$, where $\text{Tr}(d)$ is the usual field trace (this is clear by Newton's inversion formulas).
- (3) If x, y satisfy that $x^n, y^n \in F$ and $yx = \rho_n xy$ then, $(x + y)^n = x^n + y^n \in F$. Indeed one can check that $\text{Tr}((x + y)^m) = 0$ for all $1 \leq m \leq n-1$, (which is clear as $\text{Tr}(x^i y^j) = 0$ for $(i, j) \neq (0, 0) \pmod n$) thus $(x + y)^n \in F$. On the other hand when explicitly computing $(x + y)^n$ one gets $x^n + y^n + M$ where M is a sum of monomials of the form $f_{i,j} x^i y^j$ for $1 \leq i, j \leq n-1$ which are linearly independent and not in F thus we conclude that $f_{i,j} = 0$ for all $1 \leq i, j \leq n-1$ and $(x + y)^n = x^n + y^n \in F$.

Examples

Consider $A = (a, b)_n$ with standard generators x, y .

- (1) $V_1 = Fx$ and $V_2 = Fy$ are by definition one dimensional n -Kummer spaces with norms $N_1(fx) = f^n a$ and $N_2(fy) = f^n b$ respectively.
- (2) $V = Fx + Fy$ is a two dimensional n -Kummer space with norm $N_V(fx + gy) = f^n a + g^n b$.
- (3) $V = F[x]y$ is an n -dimensional n -Kummer space with norm $N_V(ky) = N_{F[x]/F}(k)b$ for $k \in F[x]$.
- (4) $V = Fx + F[x]y$ is an $n + 1$ -dimensional n -Kummer space with norm $N_V(fx + ky) = f^n a + N_{F[x]/F}(k)b$ for $k \in F[x]$.

Our objective is to find high dimensional n -Kummer spaces so that the C_m -property will ensure a non-trivial solution to the norm form. However, Example (4) is maximal with respect to inclusion ([6]) and we actually conjecture that if A is a division algebra the maximal dimension of such a space is $n + 1$. Thus we consider tensor products of symbol algebras.

Let A be the tensor product $\otimes_{i=1}^t (a_i, b_i)_n$, with the standard pairs of generators x_i, y_i for the symbol algebras. Let $V_0 = F$ and for $j = 1, \dots, t$ let $V_j \subset \otimes_{i=1}^j (a_i, b_i)_n$ be defined by

$$V_j = V_{j-1}x_j + F[x_j]y_j$$

(so in particular $V_1 = Fx_1 + F[x_1]y_1$). These are called standard n -Kummer spaces.

Every $v_k \in V_k$ defines two vectors $\tilde{v}_k \in V_1 \times \dots \times V_k$ and $\tilde{N}(v_k) \in F^k$ by setting:

$$\tilde{v}_k = (v_1, \dots, v_k)$$

such that for each i , $v_i = v_{i-1}x_i + k_i y_i$ where $k_i \in F[x_i]$ and

$$\tilde{N}(v_k) = (N_1, \dots, N_k)$$

where $N_i = N_{V_i}(v_i)$.

Proposition 3.2. *Let A and V_1, \dots, V_t be as above. Then*

- (1) $\dim(V_j) = jn + 1$.
- (2) V_j is an n -Kummer space for every $j \geq 0$.
- (3) $N_0(f) = f^n$ and for $j > 0$

$$N_{V_j}(v_{j-1}x_j + k_j y_j) = N_{V_{j-1}}(v_{j-1})a_j + N_{F[x_j]/F}(k_j)b_j$$

Proof.

(1) This is clear.

(2)+(3) For $t = 0$ it is clear.

For $t > 0$ we have for every $v_{t-1} \in V_{t-1}$ and $k_t \in F[x_t]$ that

$$\begin{aligned} N_{V_t}(v_{t-1}x_t + k_ty_t) &= (v_{t-1}x_t + k_ty_t)^n = (v_{t-1}x_t)^n + (k_ty_t)^n \\ &= v_{t-1}^n x_t^n + N_{F[x_t]/F}(k_t)y_t^n = N_{V_{t-1}}(v_{t-1})a_t + N_{F[x_t]/F}(k_t)b_t \in F. \end{aligned}$$

□

It turns out that standard n -Kummer spaces are connected to presentations of A as a tensor product of symbol algebras in the following way:

Theorem 3.3. *Let A and V_1, \dots, V_t be as above.*

- (1) *If $v_t \in V_t$ is such that $N_{V_t}(v_t) \neq 0$, then one can rewrite A as a product of t symbols with $N_{V_t}(v_t)$ as one of the slots.*
- (2) *If $v_t \in V_t$ is such that $\tilde{N}(v_t) \in F^{\times k}$ then A can be rewritten as $\otimes_{i=1}^t (N_i, *)_n$.*
- (3) *Assume $F[x_i|x_i^n = a_i]$ is a field for each i . If $N_{V_t}(v_t) = 0$ for some nonzero $v_t \in V_t$, then A can be rewritten as a product of $t - 1$ symbols.*

Proof.

- (1) For $t = 1$ write $v = cx_1 + ky_1$ for $c \in F$ and $k \in F[x_1]$. The elements $u = x_1^{-1}ky_1$ and v satisfy $uv = \rho vu$, so

$$A \cong (v^n, u^n)_n = (N_{V_1}(v), u^n)_n.$$

For $t > 1$, let $A' = (a_1, b_1)_n \otimes \dots \otimes (a_{t-1}, b_{t-1})_n$ so that $A = A' \otimes (a_t, b_t)_n$. Let $v_t \in V_t$ be such that $N_{V_t}(v_t) \neq 0$. Write $v_t = v_{t-1}x_t + k_ty_t$ for $v_{t-1} \in V_{t-1}$ and $k_t \in F[x_t]$. We know that $N_{V_t}(v_t) = v_t^n = N_{V_{t-1}}(v_{t-1})a_t + N_{F[x_t]/F}(k_t)b_t \neq 0$.

There are now two cases:

If $N_{V_{t-1}}(v_{t-1}) = 0$ then $N_{V_t}(v_t) = N_{F[x_t]/F}(k_t)b_t$, so

$$\begin{aligned} A &= A' \otimes (a_t, b_t)_n \cong A' \otimes (a_t, N_{F[x_t]/F}(k_t)b_t)_n \\ &= A' \otimes (N_{V_t}(v_t), a_t^{-1})_n, \end{aligned}$$

as claimed.

If $N_{V_{t-1}}(v_{t-1}) \neq 0$ then by induction we can write A' as a product of $t - 1$ symbols of which the last is $(N_{V_{t-1}}(v_{t-1}), d_{t-1})_n$ for

some $d_{t-1} \in F^\times$. Now apply proposition 2.2(4) to the algebra

$$(N_{V_{t-1}}(v_{t-1}), d_{t-1})_n \otimes (a_t, b_t)_n$$

writing it as a tensor product of two symbols where one of the slots is $N_{V_{t-1}}(v_{t-1})a_t + N_{F[x_t]/F}(k)b_t = N_{V_t}(v_t)$ and we are done.

- (2) This is analogous to (1), noting that in the last step where we apply proposition 2.2(4) to the algebra

$$(N(v_{t-1}), d_{t-1})_n \otimes (a_t, b_t)_n$$

we get that it is isomorphic to $(N(v_{t-1}), *)_n \otimes (N(v_t), *)_n$.

- (3) Let t' be minimal with respect to the property that N has a nontrivial zero on $V_{t'}$. Reducing the length of the product of the first t' symbols, we may assume that $t' = t$. Let A' be the product of the first $t - 1$ symbols, as before. Let $0 \neq v \in V_t$ with $N(v) = 0$. Write $v = v_{t-1}x_t + ky_t$ where $v_{t-1} \in V_{t-1}$ and $k \in F[x_t]$. If $v_{t-1} = 0$ then we would have $v_t = ky_t$. Then $0 = N(v_t) = N_{F[x_t]/F}(k)b_t$ forces $k = 0$ since $F[x_t]$ is a field, and thus $v = 0$, contrary to assumption. So we may assume $v_{t-1} \neq 0$, and then $N(v_{t-1}) \neq 0$, by minimality, so by part (1) of this proposition we may write this algebra as a product of $t - 1$ symbols where the final symbol is $(N(v_{t-1}), d_{t-1})$ for some $d_{t-1} \in F^\times$.

By 2.2(3),

$$(N(v_{t-1}), d_{t-1})_n \otimes (a_t, b_t)_n \cong (N(v_{t-1}), *)_n \otimes (N(v_{t-1})a_t, N_{F[x_t]/F}(k)b_t)_n.$$

But $(N(v_{t-1})a_t, N_{F[x_t]/F}(k)b_t)_n$ splits since

$$N(v_{t-1})a_t + N_{F[x_t]/F}(k)b_t = N(v) = 0$$

and $(c, -c)_n$ is split for every $c \in F^\times$.

□

Theorem 3.4. *Let F be a field containing all n -th roots of unity, with the property that every homogeneous equation of degree n in $f(n)$ variables has a non-trivial solution. Then every $A \in \text{Br}(F)$ of exponent n is similar to the product of at most $s = \left\lceil \frac{f(n)}{n} \right\rceil - 1$ symbols of degree n .*

Proof. We will show that every product of $s + 1$ symbols of degree n is similar to the product of s symbols of degree n and the theorem will follow after applying the Merkurjev-Suslin theorem. Let

$B = \prod_{i=1}^{s+1} (a_i, b_i)_n$. First we use 2.1, so we assume $F[x_i]$ is a field for all i . By 3.2(2) we have $V_{s+1} \subset B$, which is a linear space of dimension $(s+1)n+1$ and the norm form on it is homogeneous of degree n . But $(s+1)n+1 \geq f(n)+1 > f(n)$ thus there exist a non zero $v \in V_{s+1}$ such that $N(v) = 0$. Thus applying 3.3 we get that B is similar to the product of at most $s = \left\lceil \frac{f(n)}{n} \right\rceil - 1$ symbols.

□

Theorem 3.5. *Let F be a C_m field containing all n -th roots of unity and $A \in \text{Br}_n(F)$ be of exponent n . Then,*

$$\text{len}(n, A) \leq n^{m-1} - 1$$

that is $\text{len}(n, F) \leq n^{m-1} - 1$.

Proof. Any C_m field satisfies the property that every homogeneous equation of degree n in more than n^m variables has a solution. Thus by 3.4 every A of exponent n is similar to the product of at most $\text{len}(n, F) \leq \frac{n^m}{n} - 1 = n^{m-1} - 1$ symbols of degree n , proving the theorem. □

4. IMPROVING THE RESULT FOR NON-PRIME EXPONENT

It is a standard fact that we have a primary decomposition for the Brauer group, that is, if $\exp(A) = n = \prod_{i=1}^t p_i^{n_i}$ where p_1, \dots, p_t are different primes, then $A = \prod_{i=1}^t A_i$ where A_i is of exponent $p_i^{n_i}$. The first improvement comes from writing each of the A_i as a product of symbols.

Proposition 4.1. *Assume $(n_1, n_2) = 1$ and set $\rho_{n_1} = \rho_{n_1 n_2}^{n_2}$ and $\rho_{n_2} = \rho_{n_1 n_2}^{n_1}$. Then $(a_1, b_1)_{n_1} \otimes (a_2, b_2)_{n_2} \cong (a_1^{n_2} a_2^{n_1}, b_1^{n_2 k} b_2^{n_1 s})_{n_1 n_2}$ where*

$$s n_1 + k n_2 = 1 \pmod{n_1 n_2}.$$

Proof. Let x_i, y_i be the standard generators for $(a_i, b_i)_{n_i}$, $i = 1, 2$. Consider the elements $u = x_1 x_2, v = y_1^k y_2^s$ and compute: $u^{n_1 n_2} = a_1^{n_2} a_2^{n_1}$, $v^{n_1 n_2} = b_1^{n_2 k} b_2^{n_1 s}$ and $vu = y_1^k y_2^s x_1 x_2 = \rho_{n_1}^k x_1 y_1^k y_2^s x_2 = \rho_{n_1}^k \rho_{n_2}^s x_1 x_2 y_1^k y_2^s = \rho_{n_1 n_2}^{k n_2 + s n_1} uv = \rho_{n_1 n_2} uv$. Thus the proposition is proved. □

Corollary 4.2. *If A is as above, then $\text{len}(n, A) \leq \max\{\text{len}(p_i^{n_i}, A_i)\}$.*

Thus it is better to consider the case where $n = p^t$ for a prime p .

Next we are going to use the well known divisibility of symbol algebras to further improve the our result on symbol length.

Proposition 4.3. *"Divisibility of Symbols" [21, page 537]*

If $A = (\alpha, \beta)_s$ then $A \sim (\alpha, \beta)_{sk}^k$, assuming $\rho_{sk} \in F$.

Theorem 4.4. *Let F be a C_m field containing all n -th roots of unity.*

If $A \in \text{Br}_n(F)$ is of exponent $n = p^t$, then:

$A = \otimes_{i=1}^t C_i$ where $C_i = \otimes_{j=1}^{p^{m-1}-1} (\alpha_{i,j}, \beta_{i,j})_{p^i}$. In particular

$$\text{len}(p^t, F) \leq t(p^{m-1} - 1).$$

Proof. For $t = 1$ this is theorem 3.5 with $n = p$.

For $t = s + 1$: Let A be as above. Define $B = A^p$ and notice that B is of exponent p^s . Thus by induction $B \sim \otimes_{i=1}^s C'_i$ where

$C'_i = \otimes_{j=1}^{p^{m-1}-1} (a_{i,j}, b_{i,j})_{p^i}$. For $i = 2, \dots, s + 1$ define

$$C_i = \otimes_{j=1}^{p^{m-1}-1} (\alpha_{i,j}, \beta_{i,j})_{p^i}$$

where $\alpha_{i,j} = a_{i-1,j}$; $\beta_{i,j} = b_{i-1,j}$ and define $B' = \otimes_{i=2}^{s+1} C_i$.

Notice $B'^p = B$. Thus considering $C_1 = A \otimes B'^{-1}$, we get

$C_1^p = A^p \otimes B'^{-p} \sim B \otimes B^{-1} \sim 1$. Thus C_1 is of exponent p , and by 3.3

$C_1 \sim \otimes_{j=1}^{p^{m-1}-1} (\alpha_{1,j}, \beta_{1,j})_p$, implying

$$A \sim C_1 \otimes B' = \otimes_{i=1}^{s+1} C_i$$

where C_1, \dots, C_{s+1} are as in the theorem. □

5. FIELDS CONTAINING A C_m FIELD

In this section we consider the more general case where the base field F contains a C_m field. Notice that this class of fields includes fields such as $\mathbb{C}(x_i, y_i, i \in \mathbb{Z})$, where there is no hope of finding bounds which are a function of the exponent alone as was explained in the background section.

Thus consider a central simple algebra A , of exponent p^t and degree p^s over a field F containing a C_m field L .

The main idea is that even though F might not be finitely generated over L , A is defined over a "smaller" field K , which is finitely generated over L . This is expressed by considering the essential dimension of A over L , denoted $\text{ed}_L(A)$.

Definition 5.1. *Let A be as above. The essential dimension $\text{ed}_L(A)$ of A over L is,*

$$\min\{\text{trdeg}_F(K) \mid L \subseteq K \subseteq F \text{ and } A = A' \otimes_K F \text{ for } A' \in \text{Br}(K)\}$$

It is known that $\text{ed}_L(A)$ is finite and bounded by functions of the degree of A . For known bounds and more on the subject we refer the reader to [15] and [17]. Consider A as above. Then there exists a field $L \subseteq K \subseteq F$ with $\text{trdeg}_F(K) = \text{ed}_L(A)$ and an K -csa A' such that $A' \otimes_K F = A$. Thus if we write A' as a sum of symbols (which we can since K is a $C_{m+\text{ed}_L(A)}$ field) we can tensor up to F and write A as a product of symbols.

The only problem is that the exponent of A' can be bigger than that of A , and the resulting presentation of A will use symbols of higher degrees than needed.

In order to deal with that we need to consider a specialized essential dimension, that is

$$\text{ed}_{\text{exp}}(A) = \min\{\text{trdeg}_L(K) \mid L \subseteq K \subseteq F; A = A' \otimes_K F; A' \in \text{Br}(K); \exp(A') = \exp(A)\}.$$

Proposition 5.2. $\text{ed}_{\text{exp}}(A) \leq \text{ed}_F(A) + \dim(\text{SB}(D))$ where D is the underlying division algebra of $A'^{\otimes \exp(A)}$, A' is as in the definition of the essential dimension and $\text{SB}(D)$ is the Severi-Brauer variety of D .

Proof. It is enough to find a field $K \subseteq M \subseteq F$ such that $\exp(A'_M) = \exp(A)$ and $\text{trdeg}_L(M) \leq \text{ed}_L(A) + \dim(\text{SB}(D))$. Notice that by the definition of A' , F is a field satisfying $\exp(A'_F) = \exp(A)$. In particular $D_F \sim A'_F{}^{\otimes \exp(A)} \sim F$, so there is a rational point on $\text{SB}(D)_F$. In other words there is a specialization of the function field of $\text{SB}(D)$, $\nu : K(\text{SB}(D)) \rightarrow F$. Let $K \subseteq M \subseteq F$ be the image of ν . Clearly M satisfies all our requirements and $\text{trdeg}_L(M) \leq \text{ed}_L(A) + \dim(\text{SB}(D))$. \square

Theorem 5.3. *Let A be as above. Then the symbol length of A is bounded by $\text{len}(p^t, M)$ where M is a $C_{m+\text{ed}_L(A)+p^{s-t}-1}$ field.*

Proof. Let M be as in the proof of proposition 5.2. We want to bound the transcendence degree of M . Notice that $\dim(\text{SB}(D)) = \text{ind}(D) - 1$ thus we want to bound $\text{ind}(D)$. Since $D \sim A'^{\otimes \exp(A)}$ we have $\text{ind}(D) \leq \frac{\text{ind}(A')}{\exp(A')}$. Also, $\text{ind}(A') \leq \deg(A)$ as $A' \otimes F = A$ and by assumption $\exp(A') = \exp(A)$. Thus, $\text{ind}(D) \leq \frac{\text{ind}(A')}{\exp(A')} \leq \frac{\deg(A)}{\exp(A)} = p^{s-t}$. It follows that M is a $C_{m+\text{ed}_L(A)+p^{s-t}-1}$ field and applying theorem 4.4 we see that $\text{len}(p^s, p^t, A) \leq \text{len}(p^t, M)$ where M is as in the theorem. \square

Remark 5.4. *If we replace A by its underlying division algebra, D_A , in the above we get the bound $\text{len}(m + \text{ed}_L(D_A) + \frac{\text{ind}(A)}{\exp(A)} - 1, p^t)$, which might seem smaller as $\text{ind}(A) \leq \deg(A)$. However it might also happen that $\text{ed}_L(D_A) > \text{ed}_L(A)$. For example, if D is a generic division algebra of index 4 over a field containing ρ_4 we know by [17] that $\text{ed}(D) = 5$ where as by [15] $\text{ed}(M_2(D)) \leq 4$ since D is similar to the tensor product of two symbols of degree 4 and 2 respectively.*

6. THE BRAUER DIMENSION OF A C_m FIELD

For F a C_m field and $A \in \text{Br}(F)$ of exponent p^n , we will show that $\text{ind}(A) \leq \exp(A)^{p^{m-1}-1}$, that is $\text{Br. dim}_p(F) \leq p^{m-1} - 1$. We first reduce to the exponent p case and deduce the theorem from our previous results on symbol length.

Proposition 6.1. *Suppose F and all its algebraic extensions, L , have the property that for all central simple A/L of exponent p satisfies, $\text{ind}(A) \leq p^s$. Then, any A/F of exponent p^n satisfies, $\text{ind}(A) \leq p^{ns}$.*

Proof. Clearly the case $n = 1$ holds by assumption. Let A be of exponent p^{n+1} . Consider $B = A^p$. Then B is of exponent p^n and we have $\text{ind}(B) \leq \exp(B)^s$. Let L be a splitting field for B with $[L : F] = \text{ind}(B)$. Also consider $A_L \in \text{Br}(L)$. We have $(A_L)^p = A_L^p = B_L = 1$, thus A_L is of exponent p . Now by our assumption on F and its algebraic extensions we have that $\text{ind}(A_L) \leq p^s$. Take a splitting field $L \subset K$ of A_L with $[K : L] = \text{ind}(A_L)$ and consider K as an extension of F . Then

$$[K : F] = [K : L][L : F] = \text{ind}(B) \text{ind}(A_L) \leq p^{ns} p^s = p^{(n+1)s} \text{ and } A_K = 1.$$

Thus $\text{ind}(A) \leq p^{(n+1)s}$ as needed. \square

Proposition 6.2. *Let F be a C_m field and L be any algebraic extension of F . For every A/L of exponent p we have $\text{ind}(A) \leq p^s$.*

Proof. Since L is algebraic over F and F is C_m so is L . Now the proposition follows from Theorem 3.5. \square

Combining 6.1 and 6.2 we get:

Theorem 6.3. *If F is a C_m field, then $\text{Br. dim}_p(F) \leq p^{m-1} - 1$.*

7. THE CASE OF p -ALGEBRAS

In this section we deal with the case where F has characteristic p and $A \in \text{Br}_{p^e}(F)$. It turns out that things are much simpler in this case, both for general fields as shown by Florence in [9] and for C_m fields where things basically follow from an exercise in Serre, [25].

7.1. General fields of characteristic p .

In [9] Florence proves the following theorem:

Theorem 7.1. *If F is a field with $\text{char}(F) = p$ and A is an F -csa of index p^n and exponent p^e , then A is similar to the product of at most $p^n - 1$ cyclic algebras.*

Here is a short proof along the lines of [9]. The idea is based on the following two well known theorems.

- (1) Let A be a p -algebra of exponent p^n , then

$$F^{\frac{1}{p^n}} = F[x_f, f \in F^\times | x_f^{p^n} = f]$$

splits A ([21] page 575, exercises 30,31).

- (2) If $A \in \text{Br}(F)$ is split by a purely inseparable extension of the form $K = F[x_1, \dots, x_t | x_i^{n_i} = \alpha_i \in F^\times]$, then A is similar to the tensor product $A = \otimes_{i=1}^t A_i$, where A_i is a cyclic p -algebra with maximal subfield $K_i = F[x | x^{n_i} = \alpha_i]$, and in particular the symbol length of A is at most t (Albert, [1] theorem 28, page 108).

The idea is then to start with the splitting field $F^{\frac{1}{p^e}}$ which (in general) is infinite dimensional over F , and to find a finite dimensional subfield splitting A . Then one uses Albert's theorem to present A as the product of cyclic p -algebras.

Proof. of Theorem 7.1.

Let $\text{SB}(A)$ denote the Severi-Brauer variety of A and $F\text{SB}(A)$ its function field. Since $F^{\frac{1}{p^e}}$ splits A there is a place $\nu : F(\text{SB}(A)) \rightarrow F^{\frac{1}{p^e}}$. Let $K \subset F^{\frac{1}{p^e}}$ be the image of ν . First notice that $[K : F] < \infty$ since $\text{SB}(A)$ is finitely generated and $F^{\frac{1}{p^e}}$ is algebraic over F . It remains to bound $[K : F]$. Since dimension is invariant under scalar extensions, it is enough to bound $[K \otimes F^{sep} : F^{sep}]$, but $A \otimes F^{sep}$ is split and thus $\text{SB}(A) \times F^{sep} \cong \mathbb{P}^{(p^n-1)}(F^{sep})$, which implies

$$F^{sep}[\text{SB}(A)] \cong F^{sep}[x_1, \dots, x_{p^n-1}].$$

Thus the image of ν is generated by $p^n - 1$ elements. Now the image of every element is algebraic of degree at most p^e , which implies $[K \otimes F^{sep} : F \otimes F^{sep}] \leq p^{e(p^n-1)}$ and we are done by Albert's theorem above. \square

7.2. C_m fields. Now we assume F is a C_m field with $\text{char}(F) = p$. As we just saw above we want to bound the dimension of the image of ν . But since F is C_m , an exercise in Serre ([25] page 89 exercise 3) tells us:

Proposition 7.2. *If F is as above, then $[F^{\frac{1}{p^e}} : F] \leq p^{em}$.*

Proof. Assume we found a subfield $K \subseteq F^{\frac{1}{p^e}}$ of dimension p^{em} over F . We now show $K = F^{\frac{1}{p^e}}$. Pick a basis $\{k_1, \dots, k_{p^{em}}\}$ for K over F such that $k_i^{p^e} = \alpha_i$. Let $y \in F^{\frac{1}{p^e}}$ so that $y^{p^e} = \beta$. We will show $y \in K$. Consider the homogeneous equation

$$\sum_{i=1}^{p^{em}} x_i^{p^e} \alpha_i = x_{p^{em}+1}^{p^e} \beta.$$

Since this is of degree p^e in $p^{em} + 1 > p^{em}$ variables, the C_m property implies there is a non-trivial solution $\bar{s} = (x_1, \dots, x_{p^{em}}, x_{p^{em}+1})$. It is enough to show $x_{p^{em}+1}$ is not zero, as in this case we see the above element $t = \frac{1}{x_{p^{em}+1}} \sum_{i=1}^{p^{em}} x_i k_i \in K$ satisfies $t^{p^e} = \beta$, and thus the element $y \in F^{\frac{1}{p^e}}$ is actually in K . To see $x_{p^{em}+1} \neq 0$ assume it is zero. Then the element $u = \sum_{i=1}^{p^{em}} x_i k_i \in K$ satisfies $u^{p^e} = 0$ implying $u = 0$ so $\bar{s} = \bar{0}$ contrary to the assumption $\bar{s} \neq \bar{0}$. \square

Corollary 7.3. *Let F be as above and $A \in \text{Br}(F)$ be of exponent p^e . Then A is similar to the product of at most m cyclic p algebras of degree p^e , and in particular $\text{len}(p^e, F) \leq m$ and $\text{Br. dim}(F) \leq m$.*

8. SYMBOL LENGTH IN $K_2^M(F)/nK_2^M(F)$

In this section we observe that the basic relations we used in section 3 also hold for $K_2^M(F)/nK_2^M(F)$, and thus the main theorem can be formulated in this context. (Notice that roots of unity are not needed.)

We then show explicitly how to shorten the symbol length of an element in $K_2^M(F)/2K_2^M(F)$ over a C_2 field, which by the theorem should be one symbol. This computation illustrates the process of shortening the symbol length for $p = 2$ but clearly enables one to see

the same process will work (but will be much longer) for any prime power n .

The following relations are well known over any field F .

Proposition 8.1. *If $\{a, b\}, \{c, d\} \in K_2^M(F)/nK_2^M(F)$, then:*

- (1) $\{a, 1 - a\} = 0$ for $a \neq 1, 0$.
- (2) $\{f, 1\} = 0$ for $f \in F^\times$.
- (3) $\{a, b\} = \{f^n a, b\}$ for $f \in F^\times$.
- (4) $\{a, b\} = \{a, N_{K/F}(k)b\}$ for $K = F[\sqrt[n]{a}]$ and $k \in K^\times$.
- (5) $\{f, -f\} = 0$ for $f \in F^\times$.
- (6) If $a + b \in F^\times$, then $\{a, b\} = \{a + b, -a^{-1}b\}$.
- (7) $\{a, b\} + \{c, d\} = \{a, bd^{-1}\} + \{ac, d\}$.

Proof. (1) This is one of the defining relations of $K_2^M(F)$.

(2) This is true even in $K_2^M(F)$. Notice

$$\{f, 1\} = \{f, 1^2\} = \{f, 1\} + \{f, 1\},$$

and thus $\{f, 1\} = 0$.

- (3) Compute $\{f^n a, b\} = \{f^n, b\} + \{a, b\} = n\{f, b\} + \{a, b\} = \{a, b\}$.
- (4) Let $k \in K^\times$. Using the projection formula one computes

$$\begin{aligned} \{a, N_{K/F}(k)\} &= \text{Cor}_{K/F}(\{a, k\}) = \text{Cor}_{K/F}(\{(\sqrt[n]{a})^n, k\}) \\ &= \text{Cor}_{K/F}(n\{\sqrt[n]{a}, k\}) = 0. \end{aligned}$$

Thus $\{a, N_{K/F}(k)b\} = \{a, b\} + \{a, N_{K/F}(k)\} = \{a, b\}$.

(5) Consider $K = F[\sqrt[n]{f}]$ and compute

$$N_{K/F}(-\sqrt[n]{f}) = (-1)^n N_{K/F}(\sqrt[n]{f}) = (-1)^n (-1)^{n-1} f = -f.$$

Thus by (2) and (4) we have

$$0 = \{f, 1\} = \{f, N_{K/F}(-\sqrt[n]{f})1\} = \{f, -f\}.$$

(6) Compute

$$\begin{aligned} \{a + b, -a^{-1}b\} &= \{a(1 + a^{-1}b), -a^{-1}b\} \\ &= \{a, -a^{-1}b\} + \{1 + a^{-1}b, -a^{-1}b\} \stackrel{(1)}{=} \{a, -a^{-1}b\} \\ &= \{a, -a^{-1}\} + \{a, b\} = -\{a, -a\} + \{a, b\} \stackrel{(5)}{=} \{a, b\}. \end{aligned}$$

(7) Compute

$$\{a, bd^{-1}\} + \{ac, d\} = \{a, b\} + \{a, d^{-1}\} + \{a, d\} + \{c, d\} = \{a, b\} + \{c, d\}.$$

□

Theorem 8.2. *Let F be a C_m field and let $\alpha \in K_2^M(F)/nK_2^M(F)$, where $n = p^k$. Then α can be written as the sum of at most $k(p^{m-1} - 1)$ symbols.*

We are ready for the example. Let F be a C_2 field and let

$$\alpha = \{a_1, b_1\} + \{a_2, b_2\} \in K_2^M(F)/2K_2^M(F).$$

We will show that α can be rewritten as a single symbol as stated in the theorem. Recall the norm forms from section 2 attached to α , namely: Letting x_i, y_i be the standard generators for the two quaternions, $(a_i, b_i)_2$, $L_i = F[x_i] = F[\sqrt{a_i}]$, we define

$$V_1 = Fx_1 + L_1y_1$$

$$V_2 = V_1x_2 + L_2y_2$$

and their norm forms

$$N_1(v_1 = fx_1 + l_1y_1) = f^2a_1 + N_{L_1/F}(l_1)b_1$$

$$N_2(v_2 = v_1x_2 + l_2y_2) = N_1(v_1)a_2 + N_{L_2/F}(l_2)b_2$$

We may assume L_i 's are fields. Notice that $\deg(N_2) = 2$ and $\dim(N_2) = 5 > 2^2$, and thus there exist a non-zero $v_2 = v_1x_2 + l_2y_2 \in V_2$ where $v_1 = fx_1 + l_1y_1$, such that $N_2(v_2) = 0$. If $v_1 = 0$, we get $N_{L_2/F}(l_2)b_2 = 0$, implying $v_2 = 0$ and thus $v_1 \neq 0$. Also, if $N_1(v_1) = 0$ we have

$$\{a_1, b_1\} \stackrel{(4)}{=} \{a_1, N_{L_1/F}(l_1)b_1\} \stackrel{(3)}{=} \{f^2a_1, N_{L_1/F}(l_1)b_1\} \stackrel{N_1(v_1)=0}{=} (c, -c) \stackrel{(5)}{=} 0$$

and α is one symbol. Thus we assume $N_1(v_1) \neq 0$. From the above we write

$$\begin{aligned} \{a_1, b_1\} + \{a_2, b_2\} &= \{f^2a_1, N_{L_1/F}(l_1)b_1\} + \{a_2, b_2\} \\ &\stackrel{(6,4)}{=} \{N_1(v_1), (f^2a_1)^{-1} N_{L_1/F}(l_1)b_1\} + \{a_2, N_{L_2/F}(l_2)b_2\} \\ &\stackrel{(7)}{=} \{N_1(v_1), ((f^2a_1)^{-1} N_{L_1/F}(l_1)b_1)(N_{L_2/F}(l_2)b_2)^{-1}\} + \{N_1(v_1)a_2, N_{L_2/F}(l_2)b_2\} \\ &\stackrel{N_2(v_2)=0}{=} \{N_1(v_1), (f^2a_1)^{-1} N_{L_1/F}(l_1)b_1(N_{L_2/F}(l_2)b_2)^{-1}\} + \{c, -c\} \\ &\stackrel{(5)}{=} \{N_1(v_1), (f^2a_1)^{-1} N_{L_1/F}(l_1)b_1(N_{L_2/F}(l_2)b_2)^{-1}\}. \end{aligned}$$

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